ON THE SIGN DEFINITENESS OF QUADRATIC FORMS

(O ZNAKOOPREDELENNOSTI KVADRATICHNYKH FORM)

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We shall investigate the definiteness and sign of certain quadratic forms, which occur in the applications of the second Liapunov's method, dealing with the stability of motion.

Let the following be real quadratic forms in n variables

$$A(X, X) = \sum_{i, j=1}^{n} a_{ij}x_i x_j, \qquad B(Y, Y) = \sum_{i, j=1}^{n} b_{ij}y_i y_j$$

$$(A = \| a_{ij} \|_1^n,\ B = \| b_{ij} \|_1^n;\ A(X, X) = X'AX,\ B(Y, Y) = Y'BY)$$

where A and B are square, symmetric matrices of the coefficients, X and Y are column vectors x_i and y_i $(i = 1,\ldots,n)$, and X' and Y' are transposes of X and Y respectively [1].

Consider a quadratic form in $2n$ variables x_i and y_i

$$A(X, X) + 2kA(X, Y) + B(Y, Y) = \sum_{i, j=1}^{n} a_{ij}x_i x_j + 2k \sum_{i, j=1}^{n} a_{ij}x_i y_j + \sum_{i, j=1}^{n} b_{ij}y_i y_j \tag{1}$$

where k is a number, and the coefficient matrix of which is of the type

$$D = \begin{pmatrix} A & kA \\ kA & B \end{pmatrix} \tag{2}$$

T h e o r e m . A quadratic form in $2n$ variables (1) is positive (negative) definite if and only if quadratic forms in n variables $A(X,X)$ and $C(X,X)$ are positive (negative) definite, where the coefficient matrix C of the form $C(X,X)$ is of the type $C = B - k^2 A$.

P r o o f .

(1). The conditions are necessary.

Let (1) be positive definite; then the inequality

$$A(X, X) + 2kA(X, Y) + B(Y, Y) > 0 \tag{3}$$

holds for any values of variables, except when both are equal to zero.

For $Y = 0$, $X \neq 0$, it reduces to

$$A(X, X) > 0 \tag{4}$$

i.e. the quadratic form $A(X,X)$ is positive definite. Putting $X = \lambda Y \neq 0$ in (3), we obtain

$$\lambda^2 A(Y, Y) + 2\lambda kA(Y, Y) + B(Y, Y) > 0 \tag{5}$$

which should hold for the values $\lambda \neq 0$. This is in fact true, if

$$k^2 [A(Y, Y)]^2 - A(Y, Y) B(Y, Y) = A(Y, Y) [k^2 A(Y, Y) - B(Y, Y)] < 0 \qquad (6)$$

Hence, from (4) and (6) we obtain

$$B(Y, Y) - k^2 A(Y, Y) = C(Y, Y) > 0 \qquad (7)$$

i.e. the quadratic form $C(X,X)$ is positive definite, and consequently the conditions are necessary.

(ii). The conditions are sufficient.

We assume that the form $A(X,X)$ and $C(X,X)$ are positive definite, i.e. that (4) and (7) are valid. From (7) we have the inequality

$$A(X, X) + 2kA(X, Y) + B(Y, Y) > A(X, X) + 2kA(X, Y) + k^2 A(Y, Y) \qquad (8)$$

but

$$A(X, X) + 2kA(X, Y) + k^2 A(Y, Y) = A(X + kY, X + kY)$$

hence by the positive definiteness of $A(X,X)$, we have

$$A(X + kY, X + kY) \geqslant 0$$

and (3) follows from (8), i.e. the conditions are sufficient.

The above theorem can be formulated in a different way. The principal minors of the square symmetric matrix (2) of the order $2n$ are all positive if and only if all principal minors of the nth order square matrices A and $C = B - k^2 A$, are positive.

It is easily seen that the determinant of the matrix (2) is itself a product of the determinants of A and C, i.e.

$$\begin{vmatrix} A & kA \\ kA & B \end{vmatrix} = |A| \cdot |C|$$

Example. The necessary and sufficient conditions of positive definiteness of the quadratic form (Liapunov function) in the variables ξ_i, η_i $(i = 1,2,3)$

$$a\xi_1^2 + b\xi_2^2 + c\xi_3^2 - 2\omega (a\xi_1\eta_1 + b\xi_2\eta_2 + c\xi_3\eta_3) + 2\mu (\alpha\beta\eta_1\eta_2 + $$
$$+ \alpha\gamma\eta_1\eta_3 + \beta\gamma\eta_2\eta_3) + (\lambda + \mu\alpha^2)\eta_1^2 + (\lambda + \mu\beta^2)\eta_2^2 + (\lambda + \mu\gamma^2)\eta_3^2$$
$$(a > 0,\ b > 0,\ c > 0)$$

will represent the sufficient conditions for the stability of rotation of a heavy, rigid body [2]. The form $A(X, X) = a\xi_1^2 + b\xi_2^2 + c\xi_3^2$ is, in this case, positive definite. The matrix of the coefficients of the form $C = B - k^2 A$ $(k = -\omega)$ is of the type

$$C = \begin{vmatrix} \lambda + \mu\alpha^2 - a\omega^2 & \mu\alpha\beta & \mu\alpha\gamma \\ \mu\alpha\beta & \lambda + \mu\beta^2 - b\omega^2 & \mu\beta\gamma \\ \mu\alpha\gamma & \mu\beta\gamma & \lambda + \mu\gamma^2 - c\omega^2 \end{vmatrix}$$

and the conditions for three principal minors of C to be positive will represent the sufficient conditions for the stability.

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